PERGAMON

Available at www.MATHEMATICSweb.org
POWERED BY SCIENCE @ DIRECT®

Topology 42 (2003) 1229–1240

TOPOLOGY

www.elsevier.com/locate/top

Monodromy of real isolated singularities

Norbert A'Campo*

Universität Basel, Rheinsprung 21, CH-4051 Basel, Switzerland

Received 10 December 2001; received in revised form 13 June 2002; accepted 12 September 2002

Abstract

Complex conjugation on complex space permutes the level sets of a real polynomial function and induces involutions on level sets corresponding to real values. For isolated complex hypersurface singularities with real defining equation we show the existence of a monodromy vector field such that complex conjugation intertwines the local monodromy diffeomorphism with its inverse. In particular, it follows that the geometric monodromy is the composition of the involution induced by complex conjugation and another involution. This topological property holds for all isolated complex plane curve singularities. Using real morsifications, we compute the action of complex conjugation and of the other involution on the Milnor fiber of real plane curve singularities.

MSC: 14D05 (primary); 57M25; 14P25; 14H20; 14H50 (secondary)

Keywords: Fibered knot; Monodromy; Involution; Strong inversion; Singularity; Real morsification; Divide; Plane curve; Seifert matrix

1. Introduction

Let $f:\mathbf{C}^{n+1}\to\mathbf{C}$ be a map defined by a polynomial. We assume that $f(0)=0$ and that $0\in\mathbf{C}^{n+1}$ is an isolated critical point of f. For $p\in\mathbf{C}^{n+1}$ let $\|p\|$ denote the square root of $|z_0(p)|^2+|z_1(p)|^2+\cdots+|z_n(p)|^2$. Let $B_\varepsilon:=\{p\in\mathbf{C}^{n+1}\mid \|p\|\leqslant\varepsilon\}$, $0<\varepsilon$, be a Milnor ball for the singularity of f and let $\mathrm{Tube}_{\varepsilon,\delta}:=\{p\in B_\varepsilon\mid |f(p)|\leqslant\delta\}$, $0<\delta\ll\varepsilon$, be a regular tubular neighborhood of $\{p\in B_\varepsilon\mid f(p)=0\}$ in B_ε. A monodromy vector field X for the singularity is a smooth vector field $p\in\mathrm{Tube}_{\varepsilon,\delta}\mapsto X_p\in\mathbf{C}^{n+1}$ such that we have the following properties for $p\in\mathrm{Tube}_{\varepsilon,\delta}$ (remember $i=\sqrt{-1}$):

* Correspondence address. Langgartenstrasse 7, Biel-Benken 4105, Switzerland. Tel.: +41-61-723-0161; fax: +41-61-723-0162.

E-mail address: norbert.acampo@unibas.ch (N. A'Campo).

0040-9383/03/$ - see front matter
PII: S0040-9383(02)00099-X

- $(df)_p(X_p) = 2\pi i f(p)$,
- X_p is tangent to ∂B_ε if $p \in \partial B_\varepsilon$,
- trajectories of X starting at $p \in \partial B_\varepsilon$ are periodic with period 1 and are the boundary of a smooth disc that is transversal to f.

Using partition of unity, one can construct monodromy vector fields. The flow at time 1 of a monodromy vector field X defines a monodromy diffeomorphism $T = T_X : F \to F$, where the manifold with boundary $(F, \partial(F)) := \{p \in B_\varepsilon \mid f(p) = \delta\}$ is the Milnor fiber of the singularity. The relative isotopy class of the diffeomorphism T is independent from the chosen monodromy vector field and is the geometric monodromy of the singularity. The geometric monodromy is a topological invariant of the singularity.

From now on we will assume in addition, that the polynomial f is real meaning that its coefficients are real numbers. Let $c : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ denote the involution on complex space given by the complex conjugation of coordinate values. Hence with the above notations, we have $c(\mathrm{Tube}_{\varepsilon,\delta}) = \mathrm{Tube}_{\varepsilon,\delta}$ and $c(F) = F$. We denote by $c_F : F \to F$ the restriction of the involution c to F.

Let $X : \mathrm{Tube}_{\varepsilon,\delta} \to \mathbf{C}^{n+1}$ be a monodromy vector field for the isolated singularity of f. We may assume that we have constructed the vector field X with more care near the boundary of the Milnor ball in order to achieve that for some $\varepsilon' < \varepsilon$ we have the symmetry $c(X_p) = -c(X_{c(p)})$, $p \in \mathrm{Tube}_{\varepsilon,\delta}, \|p\| > \varepsilon'$.

Since f is real, we have

$$c((df)_p(X_p)) = (df)_{c(p)}(c(X_p)) = -2\pi i f(c(p))$$

hence, we see (by substituting q for $c(p)$ and accordingly $c(q)$ for p) that the vector field X^c defined by

$$q \in \mathrm{Tube}_{\varepsilon,\delta} \mapsto X^c_q := -c(X_{c(q)}) \in \mathbf{C}^{n+1}$$

is a monodromy vector field too. Let $Y : \mathrm{Tube}_{\varepsilon,\delta} \to \mathbf{C}^{n+1}$ be the vector field $Y := (X + X^c)/2$, which due to the extra care is also a monodromy vector field. We have $Y^c = Y$. The following is an important symmetry of the geometric monodromy.

Lemma 1. *Let T_Y be a monodromy diffeomorphism, which has been computed with a monodromy vector field Y satisfying $Y^c = Y$. We have the symmetry*

$$c_F \circ T_Y \circ c_F = T_Y^{-1}.$$

The geometric monodromy T satisfies (up to relative isotopy) the symmetry

$$c_F \circ T \circ c_F = T^{-1}.$$

Proof. The restriction of complex conjugation $c_{\mathrm{Tube}_{\varepsilon,\delta}} : \mathrm{Tube}_{\varepsilon,\delta} \to \mathrm{Tube}_{\varepsilon,\delta}$ maps the monodromy vector field Y to $-Y$ and F to F. Hence, since $c_{\mathrm{Tube}_{\varepsilon,\delta}}$ reverses the orientations of the trajectories, we have

$$T_Y^{-1} = (c_F)^{-1} \circ T_Y \circ c_F = c_F \circ T_Y \circ c_F.$$

Since the geometric monodromy T is in the relative mapping class group of the Milnor fiber represented by T_Y, for T in the relative mapping class group we have the symmetry $c_F \circ T \circ c_F = T^{-1}$. $\square$

Symmetries of monodromies as in the lemma can occur in the more general context of so-called *strongly invertible knots*, see for instance [11,17,14].

The symmetry property $c_F \circ T \circ c_F = T^{-1}$ expresses that the geometric monodromy T of a complex hypersurface with real defining equation is conjugate in the mapping class group by an element of order 2 to its inverse T^{-1}. This statement does not refer to any complex conjugation, so it can be stated for any complex hypersurface singularity with complex defining equation. We say that the singularity is *strongly invertible* if its geometric monodromy diffeomorphism T is conjugate by an element of order 2 in the relative mapping class group of the Milnor fiber to its inverse T^{-1}. The property of strong invertibility is a topological property for hypersurface singularities.

We can rewrite the symmetry property as follows: $T_Y \circ c_F \circ T_Y \circ c_F = \mathrm{Id}_F$. We see that $T_Y \circ c_F : F \to F$ is an involution of F. It follows Corollary 1.

Corollary 1. *The geometric monodromy T of an isolated complex hypersurface singularity, which is defined by a real equation, is the composition of two involutions of the fiber F_δ, $\delta \in \mathbf{R}$, namely: $T = (T \circ c_F) \circ c_F$, where c_F is the restriction of the complex conjugation.*

For $k \in \mathbf{Z}$ we also have the relation $T_Y^k \circ c_F \circ T_Y^k \circ c_F = \mathrm{Id}_F$, which shows that $T_Y^k \circ c_F : F \to F$, $k \in \mathbf{Z}$, is a sequence of involutions of F.

The above observations can be applied to plane curve singularities in general, since it follows from the Theory of Puiseux Pairs that every plane curve singularity is topologically equivalent to a singularity given by a real equation and hence plane curve singularities are strongly invertible.

For complex hypersurface singularities of higher dimension the situation seems to be opposite. In $\mathbf{C}^{n+1}$, $n > 1$, there exist isolated hypersurface singularities which are not topologically equivalent to a singularity with a real defining equation [17]. We expect that in general the geometric monodromy T of a complex hypersurface singularity fails to be strongly invertible.

For complex hypersurface singularities the eigenvalues of the monodromy T_* acting on the homology $E := \oplus_k H_k(F, \mathbf{Q})$ are roots of unity by the monodromy theorem and the characteristic polynomial of T_* is a product of cyclotomic polynomials. It follows, that the complex homological monodromy is strongly invertible. Schulze has proved strong invertibility of the homological monodromy with real coefficients [16] and the following is a strengthening of his result.

Theorem 1. *The rational homological monodromy of a complex hypersurface singularity is strongly invertible.*

Proof. Let $E = \oplus_i E_i$ a be finest possible direct sum decomposition in T_*-invariant $\mathbf{Q}$-subspaces of E. The characteristic polynomial of the restriction $A_i : E_i \to E_i$ of T_* to a summand E_i is a power $\psi(t) = \phi(t)^L$ of a cyclotomic polynomial $\phi(t)$. We have $\psi(A_i) = 0$ by the Hamilton–Cayley theorem and we have $\psi(A_i^{-1}) = 0$ since a power of cyclotomic polynomial satisfies $\psi(1/t) = \pm t^{\mathrm{degree}(\psi)}\psi(t)$.

Since the decomposition has no refinement, we may choose a vector $e_1 \in E_i$, that is cyclic for A_i and for A_i^{-1}. The systems

$$e_1, e_2 := A_i(e_1), \ldots, e_{\dim E_i} := A^{\dim E_i - 1}(e_1),$$

$$f_1 := e_1, \quad f_2 := A_i^{-1}(e_1), \ldots, f_{\dim E_i} := A^{-\dim E_i + 1}(e_1)$$

are bases for the space E_i. Let $b_i : E_i \to E_i$ be the linear map defined by $b_i(e_j) = f_j$, $1 \leqslant j \leqslant \dim E_i$.

We have $A_i^{-1}b_i(e_j)=b_iA_i(e_j),\ 1\leqslant j<\dim E_i$. The polynomial $\psi_1(t):=-\psi(t)+t^{\dim E_i}$ satisfies $\psi_1(A_i^{\pm 1})=A_i^{\pm\dim E_i}$. Since the degree of the polynomial $\psi_1(t)$ is less than $\dim E_i$, we have

$$\begin{aligned} A_i^{-1}b_i(e_{\dim E_i}) &= A^{-\dim E_i}b_i(e_1)=\psi_1(A_i^{-1})b_i(e_1)\\ &= b_i\psi_1(A_i)(e_1)=b_iA_i^{\dim E_i}(e_1)=b_iA_i(e_{\dim E_i}). \end{aligned}$$

We conclude that $A_i^{-1}b_i=b_iA_i$ and $A_i^{-1}=b_iA_ib_i^{-1}$ hold.

The polynomial $\psi_0(t):=[\psi(t)-\psi(0)]/-t\psi(0)$ satisfies $A_i^{-1}=\psi_0(A_i)$ and $A_i=\psi_0(A_i^{-1})$. We deduce $b_iA_i^{-1}=b_i\psi_0(A_i)=\psi_0(A_i^{-1})b_i=A_ib_i$ and conclude $A_i^{-1}=b_i^{-1}A_ib_i$. We observe at this point that both the conjugates of A_i by b_i and by b_i^{-1} are equal to the inverse A_i^{-1}.

For $0\leqslant j<\dim E_i$ we have (remember $e_1=f_1=b_i(e_1)=b_i(f_1)$)

$$b_i(f_{j+1})=b_iA_i^{-j}(f_1)=A_i^jb_i(f_1)=A_i^j(e_1)=e_{j+1}.$$

Hence b_i is of order two, which shows that A_i is strongly invertible over $\mathbf{Q}$. The sum $b:=\oplus b_i$ is a rational strong inversion for T_*. □

Question. Is the integral homological monodromy of a complex hypersurface singularity strongly invertible?

We would like to know the answer for the isolated surface singularity $f(x,y,z)=L_1L_2^2L_3^3L_4^4L_5^5L_6^6L_7^7L_8^8L_9^9+x^{46}+y^{46}+z^{46}$ where $L_j,\ 1\leqslant j\leqslant 9$ are linear forms on $\mathbf{C}^3$, such that the nine lines $\{L_j=0\}$ in the complex projective plane span the 9_3 configuration of flex tangents to a nonsingular cubic. No real equation for this singularity can exist, since the configuration 9_3 cannot be realized in the real projective plane, see [12,17].

In Section 1, we will study in detail the effect of complex conjugation on the topology of plane curve singularities. We show that the fiber of the link of a connected divide carries naturally a cellular decomposition with tri-valent 1-skeleton.

In Section 2, we study the involutions that appear in the decomposition of the geometric monodromy of plane curve singularities. We show that they lift to π-rotations about an axis in the universal cyclic covering of the complement of the link.

Real plane curve singularities: Let $f:\mathbf{C}^2\to\mathbf{C}$ define an isolated plane curve singularity at $0\in\mathbf{C}^2$ given by a real convergent power series $f\in\mathbf{R}\{x,y\}$. Let $f=f_1f_2\cdots f_r$ be the factorization in local branches. A factor $f_i\in\mathbf{C}\{x,y\}$ is a real convergent power series, i.e. $f_i(\mathbf{R}^2)\subset\mathbf{R}$ or the conjugate series $c(f_i)$ is a factor too.

The topology of a plane curve singularity f is completely encoded in a divide for f [4,7], see also [2,1,9].

We state the results more generally for links of connected divides, since the involution given by complex conjugation on real isolated plane curve singularities corresponds to a natural involution of links of divides as explained below, see [4,3].

A divide is the image of a generic relative immersion P of a compact one-dimensional manifold in the unit disk D in $\mathbf{R}^2$. The complement in S^3 of the link $L(P)$ of P is naturally fibered over S^1, if the divide P is connected. Let δ_P be the number of double points of the divide P. We denote the fiber surface above 1 by F_P. The local topology of a plane curve singularity is obtained from a

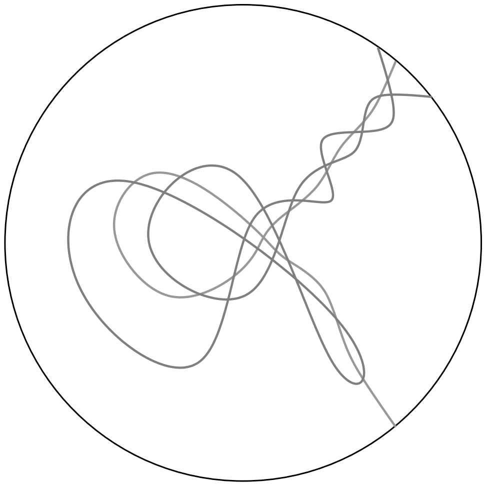


Fig. 1. Divide for the singularity $(x^3 - y^2)((x^3 - y^2)^2 - 4x^8 y)$.

divide for the singularity. More precisely, the link $L_f \subset \partial B_\varepsilon$ of an isolated plane curve singularity $\{f = 0\}, f \in \mathbf{R}\{x, y\}$, is equivalent to the link L_P of a divide P, see [4].

We recall, that for a plane curve singularity f one can obtain a divide P by performing a small real deformation f_s, $0 \leqslant s \leqslant 1$, of the singularity, where for $0 < s \leqslant 1$ the restriction of f_s to the Euclidean disk $D_\varepsilon := B_\varepsilon \cap \mathbf{R}^2$ of radius ε in $\mathbf{R}^2$ is a Morse function with $\mu(f)$ critical points and such that the 0-level is connected and contains all the saddle points. The divide P for f is the curve $\{p \in D_\varepsilon \mid f_1(p) = 0\}$, which we rescale by the factor $1/\varepsilon$ from D_ε into the unit disk D (Fig. 1).

The homology $H_1(F_P, \mathbf{Z})$ can be described combinatorially in terms of the divide P as a direct sum $H_1(F_P, \mathbf{Z}) = E_- \oplus E_0 \oplus E_+$, where E_-, E_0 and E_+ are the subspaces in $H_1(F, \mathbf{Z})$, which are freely generated as follows: $E_+ := [\delta_1, \ldots, \delta_{\mu_+}]$, $E_0 := [\delta_{\mu_+ + 1}, \ldots, \delta_{\mu_+ + \mu_0}]$ and $E_- := [\delta_{\mu_+ + \mu_0 + 1}, \ldots, \delta_{\mu_- + \mu_0 + \mu_+}]$, where $(\delta_i)_{1 \leqslant i \leqslant \mu}$ is the oriented system of vanishing cycles of the divide P with positive upper triangular Seifert form $S : H_1(F, \mathbf{Z}) \to H^1(F, \mathbf{Z}) = \mathrm{Hom}(H_1(F, \mathbf{Z}), \mathbf{Z})$, see [3, Theorem 4]. We define $N := S - \mathrm{Id}$, which is upper triangular nilpotent matrix. The monodromy $T_* : H_1(F, \mathbf{Z}) \to H_1(F, \mathbf{Z})$ is given by: $T_* = (S^t)^{-1} \circ S$.

We recall briefly the construction of the fiber surface F_P of a divide P, see [3]. Let P be a connected divide in the unit Euclidean disk D consisting of a generic system of immersed copies of S^1 and $[0, 1]$. We assume, that P is irreducible, i.e. every differentiable chord of D transversal to P, that meets P in at most 1 point, is transversally isotopic to an arc in the boundary of D. Let $f : D \to \mathbf{R}$ be a Morse function adapted to P. A region of P is a connected component of $D \setminus P$, that does not meet the boundary of D. If P has regions, we assume that the sign of f is chosen such that in at least one region the function f is positive. The link L_P of P is the subset in the 3-sphere sitting in the tangent space $S^3 := \{(x, u) \in T\mathbf{R}^2 \mid \|x\|^2 + \|u\|^2 = 1\}$ and is given by $L_P := \{(x, u) \in S^3 \mid u \in TP\}$. The number $r = r(P)$ of components of the link L_P is twice the number of immersed circles plus the number immersed intervals in P (Fig. 2).

Let S_P be a system of gradient lines of f, which connect saddle points of f with local or relative maxima or minima of f. In fact, near the boundary of D the lines of the system S_P are only

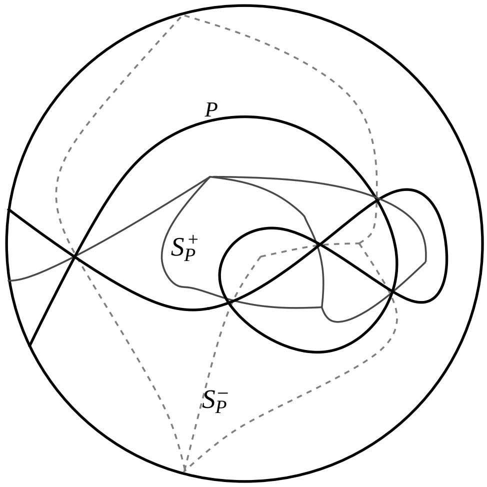


Fig. 2. Divide and graph of divide.

gradient-like, but end in a relative critical point of f. We call the set S_P the graph of the divide P. By a slalom construction, see [5], one can reconstruct from each of the sets $S_P^{\pm} := \{x \in S_P \mid \pm f(x) \geqslant 0\}$ the divide P. We assume that at the maxima and minima of f, that different arcs of S_P have different unoriented tangent directions. Let Σ_P be the following subset of the fiber F_P over 1 of the natural fibration over S^1 of the complement of the link of P: the subset Σ_P is the closure in the tangent space TD of D of

$$\Sigma'_P := \{(x,u) \in TD \mid x \in S_P, f(x) > 0, (df)_x(u) = 0, \|x\|^2 + \|u\|^2 = 1\}.$$

We recall that the union of fiber F_P with the link L_P is the closure in TD of

$$F'_P := \{(x,u) \in TD \mid x \in D, f(x) > 0, (df)_x(u) = 0, \|x\|^2 + \|u\|^2 = 1\},$$

which is a surface of genus $g(P)$ equal to the number of double points in P.

The system of cycles δ_i, $1 \leqslant i \leqslant \mu$, can be drawn on Σ_P as follows. The vanishing cycle $\delta_i, \mu_- + \mu_0 < i \leqslant \mu$, which corresponds to a local maximum M of f is the circle $\{(M,u) \mid u \in T_M D, \|M\|^2 + \|u\|^2 = 1\}$ oriented counter clock-wise. The vanishing cycle $\delta_i, \mu_- < i \leqslant \mu_- + \mu_0$, which corresponds to a saddle point z of f is a curve in the set $E_z := \{(x,u) \in TD \mid x \in e_z, u \in \mathrm{Kernel}(df)_{e_z}, \|x\|^2 + \|u\|^2 = 1\}$, as drawn in Fig. 3. The curve δ_i is a piecewise smooth embedded copy of S^1 in F_P with image in E_z and with nonconstant projection to the disk D. The orientation is chosen such that at both ends the orientation agrees with the oriented vanishing cycle of the maximum. Moreover, the inward tangent vectors (x,u) to e_z at end points $x \in e_z$ do belong to δ_i. The vanishing cycle δ_i, $1 \leqslant i \leqslant \mu_-$, which corresponds to a local minimum of f projects to an oriented circuit $e_1, e_2, \ldots, e_k$ of edges of the graph S_P. The circuit surrounds the region in counter clock-sense to which the minimum corresponds. The circuit is a polygon and bounds a cell in D. The curve δ_i is a subset of $\{(x,u) \in TD \mid x \in \bigcup_{1 \leqslant j \leqslant k} e_j, u \in \mathrm{Kernel}(df)_x, \|x\|^2 + \|u\|^2 = 1\}$ and is the image of a piecewise smooth embedding of S^1 as drawn in Fig. 4. The vectors (x,u), which belong to δ_i, point out of the cell of the circuit. Smooth representatives for the system of vanishing cycles can be obtained using tears as in [6].

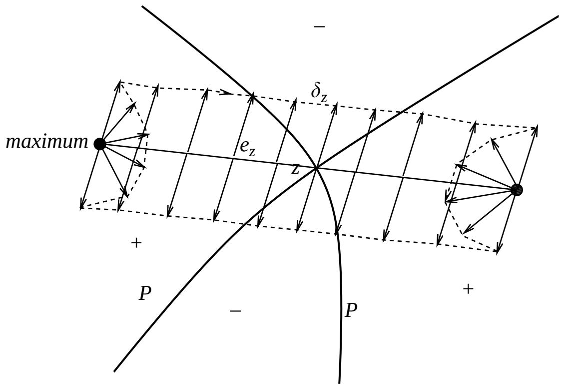


Fig. 3. Vanishing cycle δ_z for a saddle point.

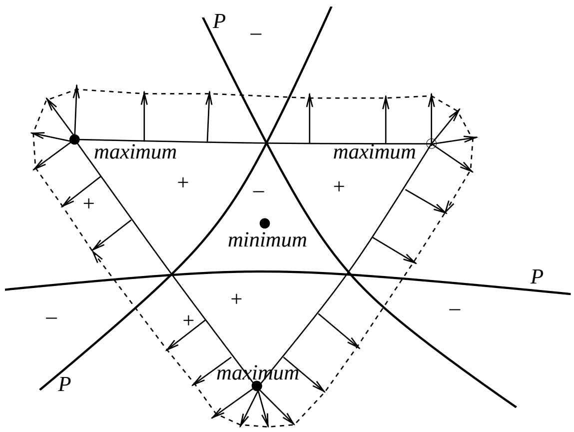


Fig. 4. Vanishing cycle δ_i for a minimum.

The involution $c_{T\mathbf{R}^2} : (x,u) \in T\mathbf{R}^2 \mapsto (x,-u) \in T\mathbf{R}^2$ induces an involution c_{S^3} on $S^3 \subset T\mathbf{R}^2$ that preserves the link L_P of any divide P and that induces involutions on the fiber surfaces above ± 1. If the divide P is a divide for a real plane curve singularity f, the involutions induced by $c_{T\mathbf{R}^2}$ or by the complex conjugation c on the triples (S^3, L_P, F_P) and on $(\partial B_\varepsilon, L_f, F)$ correspond to each other by the homeomorphism of pairs of the main theorem of [4].

Theorem 2. *Let $P \subset D$ be a connected divide consisting of the image of a relative generic immersion of a compact one-dimensional manifold in the unit disk $D \subset \mathbf{R}^2$. Let E_-, E_0 and E_+ be the summands in $H_1(F, \mathbf{Z})$ as above. The involution $c_{F*} : H_1(F, \mathbf{Z}) \to H_1(F, \mathbf{Z})$ fixes pointwise the summand E_+, in particular for $\delta_i \in E_+$ we have*

$$c_{F*}(\delta_i) = \delta_i.$$

For $\delta_i \in E_0$ we have

$$c_{F*}(\delta_i) = -\delta_i + \sum_{1 \leqslant j \leqslant \mu_+} \langle N(\delta_j), \delta_i \rangle.$$

For $\delta_i \in E_-$ we have

$$c_{F*}(\delta_i) = \delta_i + \sum_{1 \leqslant j \leqslant \mu_+} \langle N(\delta_j), \delta_i \rangle - \sum_{\mu_+ < j \leqslant \mu_+ + \mu_0} \langle N(\delta_j), \delta_i \rangle \delta_j.$$

The trace of the involution $c_{F*} : H_1(F, \mathbf{Z}) \to H_1(F, \mathbf{Z})$ is given by

$$\mathrm{Trace}(c_{F*}) = \mu_- - \mu_0 + \mu_+.$$

Proof. A vanishing cycle $\delta_i \in E_+$ corresponds to a maximum $M \in D$ of f, hence as set we have $\delta_i = \{(M, u) \in TD \mid \|M\|^2 + \|u\|^2 = 1\}$. The involution $(x, u) \mapsto (x, -u)$ induces on δ_i the antipodal map, which is orientation preserving. It follows $c_{F*}(\delta_i) = \delta_i$ in homology. A vanishing cycle $\delta_i \in E_0$ corresponds to a saddle point $z \in D$ of f. Working with the tear model of [6], see also Section 2, we see that the involution reverses the orientation and that at the endpoints, which are maxima of f, we have outward instead of inward vectors. Hence,

$$c_{F*}(\delta_i) = -\delta_i + \sum m_{i,j} \delta_j,$$

where in the sum j runs through the maxima of f in the interior of D which are connected by gradient lines of S_P to the saddle point i. The coefficient $m_{i,j}$ equals 1 or 2 depending on whether the connection by gradient lines is simple or double. Finally one gets $c_{F*}(\delta_i) = -\delta_i + \sum_{1 \leqslant j \leqslant \mu_+} \langle N(\delta_j), \delta_i \rangle$. For a vanishing cycle $\delta_i \in E_-$ one can work with the model of [6] and get $c_{F*}(\delta_i) = \delta_i + \sum_{1 \leqslant j \leqslant \mu_+} \langle N(\delta_j), \delta_i \rangle - \sum_{\mu_+ < j \leqslant \mu_+ \mu_0} \langle N(\delta_j), \delta_i \rangle$. Since N is strictly upper triangular, the matrix of c_{F*} is upper triangular with ± 1 on the diagonal. We get $\mathrm{Trace}(c_{F*}) = \mu_- - \mu_0 + \mu_+$. □

We can present the Seifert form and homological monodromy with block-matrices S, T as in [10]. On $H_1(F, \mathbf{Z}), E_-, E_0, E_+$ and $H^1(F, \mathbf{Z})$ we work with the basis or dual basis given by the system $\delta_i,\ 1 \leqslant i \leqslant \mu$. The matrix in block form of the Seifert form is

$$S = \begin{pmatrix} \mathrm{Id}_{\mu_+} & A & G \\ O & \mathrm{Id}_{\mu_0} & B \\ O & O & \mathrm{Id}_{\mu_-} \end{pmatrix},$$

where the block G equals the block matrix product $\frac{1}{2}(A \circ B)$. The matrix coefficients of $A \circ B$ and G have interpretations in terms of the divide P or in terms of the Morse function f_P on the disk D. The matrix coefficient $(A \circ B)_{(i,j)},\ 1 \leqslant i \leqslant \mu_+,\ \mu_+ + \mu_0 < j \leqslant \mu$, counts the number of sector adjacencies, that has the $+$-region i of P with the $-$-region j, while the coefficient $G_{(i,j)}$ counts the number of common boundary segments of the regions i and j. The coefficient $(A \circ B)_{(i,j)}$ also counts the number of saddle connections via gradient lines of f_P in between the minimum j and maximum i, while the coefficient $G_{(i,j)}$ counts the number of components of regular gradient line connections from the minimum j to the maximum i. This explains the above factor $\frac{1}{2}$, since a segment of P is twice incident with a saddle point of f_P.

The matrix of the action of complex conjugation on $H_1(F, \mathbf{Z})$ is

$$C = \begin{pmatrix} \mathrm{Id}_{\mu_+} & A & G \\ O & -\mathrm{Id}_{\mu_0} & -B \\ O & O & \mathrm{Id}_{\mu_-} \end{pmatrix}$$

and is obtained from the matrix S by multiplying the middle row of blocks by -1. It is interesting to compute the matrix of the involution $T \circ c_F$ on $H_1(F, \mathbf{Z})$

$$TC = \begin{pmatrix} \mathrm{Id}_{\mu_+} & O & O \\ -{}^tA & -\mathrm{Id}_{\mu_0} & O \\ {}^tG & {}^tB & \mathrm{Id}_{\mu_-} \end{pmatrix}.$$

The matrix $T \circ C$ is the transgradient of the matrix of C.

It turns out that the combinatorial property $G = \frac{1}{2}(A \circ B)$ for divides is equivalent to $C \circ C = \mathrm{Id}_\mu$ or $T \circ C \circ T \circ C = \mathrm{Id}_\mu$.

For an isolated plane curve singularity at $0 \in \mathbf{C}^2$, which is given by a real equation $\{f = 0\}, f \in \mathbf{R}\{x, y\}, f(0) = 0$, we will denote by $\delta_{\mathbf{R}}(f)$ the number of double points of a divide P for the singularity. Hence $\delta_{\mathbf{R}}(f)$ is the maximal number of local real saddle points in some level, that can occur for a small real deformation of f. Observe, that one has $\delta_{\mathbf{R}}(f) \leqslant \delta(f)$, where $\delta(f)$ is the maximal number of local critical points in some level, that can occur for a small deformation of f. We recall the formula $\mu(f) = 2\delta(f) - r + 1$ of Milnor [15]. As example for $f = x^4 + Kx^2y^2 + y^4,\ -2 \neq K \neq 2$, one has $\delta_{\mathbf{R}}(f) = 4,\ \delta(f) = 6,\ r = 4$ and very surprisingly, Callagan shows that for $-2 < K < 2$ there exists a small real deformation with five local minima in the same level [8].

Theorem 3. *For an isolated plane curve singularity at* $0 \in \mathbf{C}^2$, *which is given by a real equation* $\{f = 0\}, f \in \mathbf{R}\{x, y\}, f(0) = 0$, *we have*

$$\mu(f) = 2\delta_{\mathbf{R}}(f) + \mathrm{Trace}(c_{F*}).$$

Proof. We have $\delta_{\mathbf{R}} = \dim E_0 = \mu_0$. Hence $\mu(f) = \mu_- + \mu_0 + \mu_+ = \mathrm{Trace}(c_{F*}) + 2\mu_0 = 2\delta_{\mathbf{R}}(f) + \mathrm{Trace}(c_{F*})$. □

Curves δ_i that correspond to maxima of f_P are invariant by the involution c. A curve δ_i that corresponds to a saddle point or minimum of f_P is in general not invariant by c. The union Σ_P of the curves δ_i is invariant by c. We get Theorem 4.

Theorem 4. *Let P be a connected divide. The pair (F_P, Σ_P) defines a tri-valent cellular decomposition with $r(P)$ punctured cells of the fiber surface with boundary F_P. The involution $c : (x, u) \to (x, -u)$ acts on the map (F_P, Σ_P).*

Proof. The union Σ_P of the vanishing cycles δ_i is a tri-valent graph Σ_P that is invariant by c. The inclusion $\Sigma_P \subset F_P$ induces an isomorphism $H_1(\Sigma_P, \mathbf{Z}) \to H_1(F_P, \mathbf{Z})$, hence, Σ_P is a spine for the surface F_P. □

In particular if $r(P)=1$ the triple (F_P,Σ_P,c) is a so-called maximal unicell map of genus $g(P)$ with orientation reversing involution c with $r(P)=1$ fixed points on the graph of the map. The involution c_F has a unique fixed point on Σ_P, which corresponds to the intersection of the folding curve $F_P\cap\partial D$ of c_F with the graph Σ_P. Maximal here means that the number of edges of the graph of the map is maximal, i.e. $3g$.

It would be interesting to compute the generating series

$$\mathrm{divide}(t):=\sum_{g\in\mathbf{N}}d(g)t^g=1+t+2t^2+8t^3+36t^4+\cdots,$$

where $d(g)$ denotes the number of simple, relative and generic immersions the interval $[0,1]$ with g double points in the disk D, counted up to homeomorphism in the source and image. We have taken the coefficients up to the term t^4 of divide(t) from the listings of simple, relative, free or oriented divides by Ishikawa [13]. We ask to compare the numbers $d(g)$ with the numbers $m^+(g)$ of maximal maps of genus g with orientation reversing involution having one fixed point on its graph.

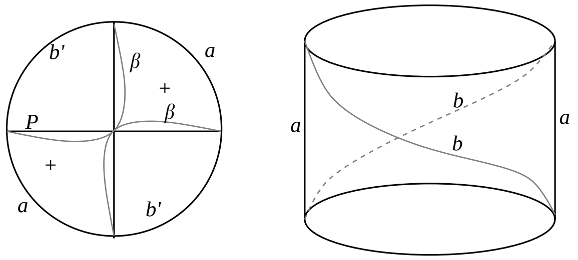
2. Involutions induced by π-rotations

Our next goal is to visualize the two involutions c_F and $T\circ c_F$. We assume that the monodromy diffeomorphism T is chosen in its isotopy class such that $T\circ c_F$ is an involution of the fiber surface F_P. In fact, we assume that the monodromy T was given by a monodromy vector field X, which satisfies $X=X^c$.

We assume for simplicity, that the divide P meets the boundary of the disk D. The involution c_F is an orientation reversing diffeomorphism of $F_1=F_P$, which fixes pointwise the system of arcs $a:=F_P\cap\partial D$. The Lefschetz number of the orientation reversing involution $T\circ c_F$ is equal to the Lefschetz number of c_F by Theorem 1. Hence the involution $T\circ c_F$ also fixes pointwise exactly a system of arcs b on the fiber surface F_P with $\chi(a)=\chi(b)$, i.e. the systems a and b consist of the same number of arcs.

Let $z:Z\to S^3\setminus K_P$ be the infinite cyclic covering of the knot complement. Let X^Z be the lift of the vector field X. Let $T_{1/2}:Z\to Z$ be the flow diffeomorphism of the vector field X^Z with stopping time $\frac{1}{2}$. Let $F_1'\subset Z$ be a lift of the fiber surface F_1 and let F_{-1}' be the lift $T_{1/2}(F_1')$ of the fiber surface F_{-1}. The involution c_F can first be lifted unambiguously to an involution of F_1' and then extended unambiguously to an involution A of Z such that A maps the vector field X^Z to its opposite. The involution A is a π-rotation about a lift of the system of arcs a into F_1'. The involution $(x,u)\mapsto(x,-u)$ induces an involution $c_{F_{-1}}$ of the fiber surface F_{-1} above -1, and, as we have done for the involution c_F, the involution $c_{F_{-1}}$ lifts unambiguously to a π-rotation $B:Z\to Z$ about a system of arcs $b''\subset F_{-1}'$ that satisfies $z(b'')=b':=\partial D\cap F_{-1}$. We have $B\circ A=T_{1/2}\circ T_{1/2}=T_1$. Since T_1 is a lift of the monodromy, it follows from $(T\circ c_F)\circ c_F=T$, that the involution $T\circ c_F$ fixes the system of arcs $b:=z(T_{1/2}^{-1}(b''))$.

In [3] we have computed the monodromy diffeomorphism $T_+:F_1\to F_{-1}$ for which $T_{1/2}$ is a lift to Z as a product of half Dehn twists for a divide. From the above we deduce, that the involution $T\circ c_F:F_1\to F_1$ is the composition $T_+^{-1}\circ c_{F_{-1}}\circ T_+$. We see that $T\circ c_F$ fixes the arc $b=T_+^{-1}(\partial D\cap F_{-1})$. We also see that the involution $c_F\circ T$ of F_1 fixes the system of arcs $c_F(b)=T_+(\partial D\cap F_{-1})$. Both arc systems b and $c_F(b)$ have equal projections β into D. See Figs. 5, 6 for examples.



Divide for $xy = 0$ Fiber surface with arc systems a,b

Fig. 5. Dehn twist as composition of two involutions.

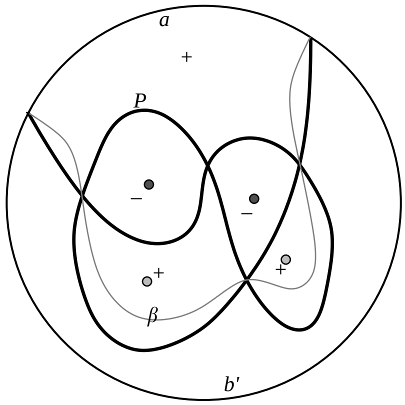


Fig. 6. Divide for the E_8 singularity and the curve β.

The composition of the orientation reversing involutions, that fix pointwise the arc systems a, b of Fig. 5 on the cylinder surface, is indeed a Dehn twist. To see this, think of the cylinder as $[0, \pi] \times S^1$; the two involutions induce on the circle $\{\theta\} \times S^1$ reflections about diameters that make an angle θ, so the composition is a rotation of angle 2θ of the circle $\{\theta\} \times S^1$.

In Fig. 6 the curve β is drawn for a more complicated divide for the plane curve singularity E_8 with equation $y^3 - x^5 = 0$.

Acknowledgements

I'd like to thank Makoto Sakuma for explaining the work of Jeffrey L. Tollefson [18] to me.

I'd like to thank the referee for having attracted my attention to work of Gusein-Zade [10], in which he shows among other results, that the integral homological monodromy of an isolated complex hypersurface singularity with real defining equation f is the composition of two involutions, both conjugated to the action of complex conjugation on the homology of a real regular fiber.

References

[1] N. A'Campo, Le Groupe de Monodromie du Déploiement des Singularités Isolées de Courbes Planes II, Actes du Congrès International des Mathématiciens, Vancouver, 1974, pp. 395–404.

[2] N. A'Campo, Le Groupe de Monodromie du Déploiement des Singularités Isolées de Courbes Planes I, Math. Ann. 213 (1975) 1–32.

[3] N. A'Campo, Generic immersions of curves, knots, monodromy and Gordian number, Publ. Math. I.H.E.S. 88 (1998) 151–169.

[4] N. A'Campo, Real deformations and complex topology of plane curve singularities, Ann. Faculté Sci. Toulouse 8 (1999) 15–23;
N. A'Campo, Real deformations and complex topology of plane curve singularities, Ann. Faculté Sci. Toulouse 2 (1999) 343.

[5] N. A'Campo, Planar trees, slalom curves and hyperbolic knots, Publ. Math. I.H.E.S. 88 (1998) 171–180.

[6] N. A'Campo, Quadratic vanishing cycles, reduction curves and reduction of the monodromy group of plane curve singularities, Tohoku Math. J. 53 (2001) 533–552.

[7] L. Balke, R. Kaenders, On certain type of Coxeter–Dynkin diagrams of plane curve singularities, Topology 35 (1995) 39–54.

[8] J. Callahan, The double cusp has five minima, Math. Proc. Cambridge Philos. Soc. 84 (3) (1978) 537–538.

[9] S.M. Gusein-Zade, Matrices d'intersections pour certaines singularités de fonctions de 2 variables, Funkcional. Anal. i Prilozen 8 (1974) 11–15.

[10] S.M. Gusein-Zade, Index of a singular point of a gradient vector field, Funkcional. Anal. i Prilozen 18 (1984) 6–10.

[11] W. Heil, J.L. Tollefson, Deforming homotopy involutions of 3-manifolds to involutions, Topology 17 (4) (1978) 349–365.

[12] D. Hilbert, S. Cohn-Vossen, Anschauliche Geometrie, Springer, Berlin, 1932 (Reprint: Dover, New York, 1944. English transl.: Geometry and the Imagination, Chelsea, New York, 1952).

[13] M. Ishikawa, Links of plane curves, fibredness, quasipositivity and Jones polynomials, Inauguraldissertation, Universität Basel, 2001.

[14] A. Kawauchi, A survey of Knot Theory, Birkhäuser, Basel, 1996.

[15] J. Milnor, Singular points on complex hypersurfaces, Ann. Math. Stud, Princeton University Press, Princeton, NJ, 1968.

[16] M. Schulze, Email communications, November 2001 and Thesis at Kaiserslautern in progress.

[17] B. Teissier, Un exemple de classe d'équisingularité irrationelle, C. R. Acad. Sci. Paris 311 (2) (1990) 111–113.

[18] J.L. Tollefson, Periodic homeomorphisms of 3-manifolds fibered over S^1, Trans. Amer. Math. Soc. 223 (1976) 223–234 (with Erratum Trans. Amer. Math. Soc. 243 (1978) 309–310).